

Problem sheet 4

Due Friday 24 April 2026, 16:00. Hand in problems 1 to 4. Problem 5 is for discussion in the tutorial.

Show your working. Where a result from the lectures is used, name it. A correct answer without a derivation earns no marks¹.

Problem 1: the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

1. Find the eigenvalues λ_1, λ_2 of A by solving $\det(A - \lambda I) = 0$.
2. Find an eigenvector for each eigenvalue and check that they are orthogonal.
3. Write $A = Q\Lambda Q^T$ with Q orthogonal, and use it to compute A^{10} without multiplying matrices ten times.

Problem 2: a definite integral

Evaluate $I = \int_0^1 \frac{\ln(1+x)}{1+x^2} dx$.

Tip

Substitute $x = \tan \theta$ first. You should reach an integral of the form $\int_0^{\pi/4} \ln(1 + \tan \theta) d\theta$, then use the symmetry $\theta \mapsto \frac{\pi}{4} - \theta$.

Problem 3: a series

Decide whether each series converges, and name the test you used.

Series	Converges?	Test
$\sum_{n=1}^{\infty} \frac{1}{n^2 + n}$		
$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$		
$\sum_{n=1}^{\infty} \frac{n!}{n^n}$		
$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$		

Problem 4: a differential equation

A tank holds $V = 500$ liters of brine. Fresh water flows in at $q = 5$ liters a minute and the mixture leaves at the same rate. With $m(t)$ the mass of salt in kilograms,

$$\frac{dm}{dt} = -\frac{q}{V} m, \quad m(0) = m_0.$$

¹ Marking scheme: 10 marks per problem, of which at most 2 for the final numerical answer. Partial credit is given for a correct method with an arithmetic slip.

1. Solve for $m(t)$.
2. After how long is $m(t) = m_0 / 2$? Give the answer in minutes to one decimal place.

Problem 5, for the tutorial: Euler's identity

Prove that $e^{i\pi} + 1 = 0$ from the Taylor series of e^x , $\sin x$ and $\cos x$.

Worked example from the lecture

The eigenvalues of $B = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$:

$$\begin{aligned}\det(B - \lambda I) &= \begin{vmatrix} 4 - \lambda & 1 \\ 2 & 3 - \lambda \end{vmatrix} \\ &= (4 - \lambda)(3 - \lambda) - 2 \\ &= \lambda^2 - 7\lambda + 10 \\ &= (\lambda - 2)(\lambda - 5)\end{aligned}$$

$$\boxed{\lambda_1 = 2, \quad \lambda_2 = 5}$$

Check: $\lambda_1 + \lambda_2 = 7 = \text{tr } B$ and $\lambda_1 \lambda_2 = 10 = \det B$. ✓

- ☐ Problem 1 handed in
- ☐ Problem 2 handed in
- ☐ Problems 3 and 4 handed in